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Mass and magnetic moment of localised electrons near conductors

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Abstract. The calculation of renormalised one-loop corrections to the anomalous magnetic moment and mass of an electron localised between conducting plates is performed within a consistent long-distance expansion. Application of the Euler-MacLaurin formula yields gauge- and cutoff-independent corrections to physical quantities in a transparent way. The boundary contributions to the wavefunction renormalisation and the vertex correction respect the Ward identity of QED. In the limit of the distance to one of the plates going to infinity we obtain the correction for a single plate at distance d. The coincidence of the corrections $\Delta a_e = \Delta m/m = -\alpha/4md$ to the electron mass m and anomaly factor a_e explains an apparent discrepancy in the recent literature.

1. Introduction

A long time ago Casimir [1] discovered an amazing consequence of the quantised vacuum structure of QED: boundary conditions constraining the quantised electromagnetic field at the surface of conductors diminish the zero-point energy and lead to an attractive force between conducting plates. In fact, all radiative corrections to QED processes are (slightly) modified near conductors. In a fundamental series of papers Babiker, Barton and Grotch [2-4] have investigated such corrections to the anomalous magnetic moment of electrons bound in hydrogen-like atoms or interacting with an external magnetic field. Their main interest was the applicability of the non-relativistic approach to magnetic effects such as the anomalous magnetic moment. Recently this subject found renewed interest for two reasons: first, Fischbach and Nakagawa [5, 6] and Kreuzer and Svozil [7, 8] tried to give a consistent formulation of QED between conducting plates and to compute the resulting modifications of the physical quantities electron mass m and anomalous magnetic moment a_e. At the same time, remarkable progress in the UC-Seattle geonium experiment [9] required a thorough treatment of systematic errors in this high-precision measurement of a_e due to the presence of conductors [10-12].

This paper is addressed to the first reason. In § 2, based upon the work of Kreuzer and Svozil [8], we recall the photon propagator, fulfilling the boundary conditions imposed by two parallel ideal conducting plates. Our approach to a consistent long-distance approximation for boundary corrections to QED processes employs fermion

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fields localised as Gaussian wavepackets. Renormalisation proceeds through subtraction of the infinite-space counterterms.

In § 3 the finite and gauge-independent results for physical quantities such as the electron mass and anomalous magnetic moment are obtained using the Euler-MacLaurin formula. We also check the Ward identity $Z_1 = Z_2$, which guarantees that there are no corrections to the gyromagnetic ratio 2 associated with the photon vertex γ_μ.

In § 4 the results are summarised in the context of recent literature on this subject. The coincidence of our findings for the mass and anomaly factor corrections, respectively, resolves a hitherto unsettled controversy. In the limit of the distance to one of the plates going to infinity we also obtain the results in the case of a single conducting plate.

2. Radiative corrections between conducting plates

In our treatment of QED processes near conductors we confine ourselves to plane conductors, where the photon propagator is easily calculated by the method of mirror charges. Electrons are assumed to be well localised, so that their contact interaction with conductors can be disregarded.

A major difficulty in obtaining the photon propagator is that in QED the electromagnetic field is usually described by the gauge potential A_μ, whereas boundary conditions constrain the gauge-invariant field strength

$$F_{\mu\nu} n_\alpha \varepsilon^{\mu\nu\alpha\beta}|_S = 0 \tag{2.1}$$

(the subscript S means 'at the surface of the conductor', and $n_0 = 0$, $\boldsymbol{n}$ is orthogonal to the surface in the rest frame of the conductor). Boundary conditions on A_μ depend on the gauge choice. In the case of a plane conductor there is a (unique translation-invariant) gauge consistent with $A|_S = 0$, namely the axial gauge $nA = 0$ with the constant vector n of (2.1). However, in order to have better control of non-covariant terms, arising due to broken translational and rotational invariance, we work in a covariant gauge at the expense of additional sign factors in the photon propagator, stemming from mirror charges with an odd number of reflections (see Kreuzer and Svozil [8] and Brown and Macley [1]). We take the conducting surfaces to be described by the planes $x^3 = d$ and $x^3 = d - a$. Then, in natural units $\hbar = c = 1$, the photon propagator in the Feynman gauge is represented by the mode sum (see figure 1)

$$\Delta_{\mu\nu}(x, y) = \frac{1}{2a} \sum_{\substack{n=-\infty \\ k_3 = n\pi/a}}^{\infty} \int \frac{\mathrm{d}^3 \underline{k}}{(2\pi)^3} \frac{-1}{k^2 + \mathrm{i}\varepsilon} \{g_{\mu\nu} \exp[\mathrm{i}k(x-y)] - \theta^n g_{\mu\bar{\nu}} \exp[\mathrm{i}k(x-\bar{y})]\} \tag{2.2}$$

with

$$\theta \equiv \exp(2\pi \mathrm{i} t) \qquad t \equiv d/a$$

and the notation for a vector v

$$\underline{v} \equiv (v^0, v^1, v^2) \qquad \bar{v}^\mu \equiv v^{\bar{\mu}} \equiv (v^0, v^1, v^2, -v^3)$$

(the sign changes of three-components of tensors according to bars do not disturb the Einstein summation convention). The mode sum representation (2.2) is related to the

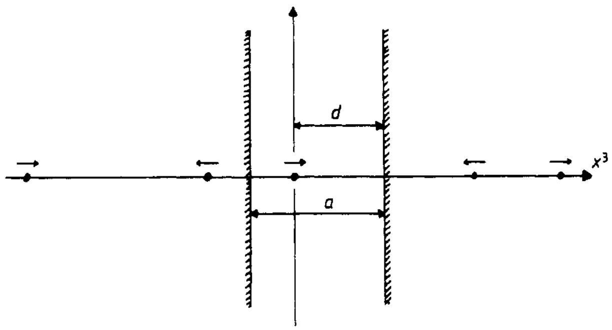

Figure 1. An electron is localised (with width σ) at the origin between conducting plates at $x^3 = d$ and $x^3 = d - a$. Mirror charges corresponding to an odd number of reflections are denoted by $\leftarrow$ and contribute sign factors due to their reversed motion perpendicular to the plates.

sum over mirror charges by Poisson's formula. The two-plates configuration of figure 1 has the advantage of eventually also yielding the one-plate results in the limit $a \to \infty$.

The renormalisation programme is carried out by subtracting from the divergent self-energy and vertex correction graphs (at one loop, see figure 2) the usual infinite-space QED counterterms. In this way we obtain finite corrected values of the electron's mass m and anomaly factor $a_e \equiv g/2 - 1$. Due to broken translational invariance we have to work in configuration space (nevertheless, it is convenient to use the Fourier integral representation of wavefunctions and propagators). In particular, it is essential to specify the location of the fields. We assume the in and out electron states to be Gaussian wavepackets localised at $x_3 = 0$ with width $\sigma \ll d, a - d$,

$$\begin{aligned} u(x) &= (2\sigma\pi^{1/2})^{1/2} \int \frac{\mathrm{d}p_3}{2\pi} \exp(-\tfrac{1}{2}\sigma^2 p_3^2 - \mathrm{i}px) u(p) \\ \bar{u}(y) &= (2\sigma\pi^{1/2})^{1/2} \int \frac{\mathrm{d}p_3'}{2\pi} \bar{u}(p') \exp(-\tfrac{1}{2}\sigma^2 p_3'^2 - \mathrm{i}p'y) \end{aligned} \tag{2.3}$$

(with normalisation $\langle \bar{u} | u \rangle = (2\pi)^3 \delta^3(\underline{p}' - \underline{p})$). This amounts to a restriction to motion parallel to the plates as will be discussed below. In order to get an impression of the magnitudes involved we quote some parameters of recent Penning trap experiments [9]. There $a \sim 1$ cm and electrons are localised by an electric quadrupole field with an electrostatic potential of ~ 10 V so that at the temperature of liquid helium $\sigma \sim 3 \times 10^{-3}$ cm. Due to the exponential fall off of the probability density in the Gaussian wavepackets (and due to the smallness of the Compton wavelength $1/m \ll \sigma$) contact interactions of electrons with the conductor (and boundary conditions on their propagator) are safely neglected.

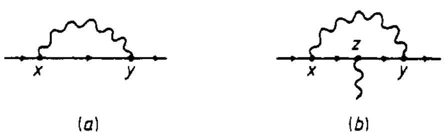

Figure 2. One-loop contributions to (a) self-energy $\Sigma(x, y)$ and (b) vertex correction $\Lambda_\mu(x, y; z)$.

We proceed in a long-distance expansion $a \gg 1/m$. With the above parameters

$$h \equiv \frac{\pi}{ma} \sim 3 \times 10^{-10}.$$

Thus, in a reasonable experimental situation the first term of an expansion in h will usually be sufficient.

Although there exists an integral representation of $\Delta(x, y)$ (Bordag *et al* [1]), obeying the boundary conditions at the plates, we make use of the mode sum representation (2.2). Boundary corrections then result in calculating the difference between mode sum and (free space) mode integral

$$D \equiv \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} \mathrm{d}n \right) f(n) \tag{2.4}$$

of some function $f(n)$. To this end the Euler-MacLaurin formula [13] (EMF)

$$h \sum_{j=0}^{n-1} f(a+jh) - \int_a^b f(x)\,\mathrm{d}x = \left(-\tfrac{1}{2} h f(t) + \sum_{l=1}^{m} h^{2l} \frac{B_{2l}}{(2l)} f^{(2l-1)}(t) \right)\Bigg|_{t=a}^{t=b} + \mathrm{O}(h^{2m+2}) \tag{2.5}$$

with $h = (b-a)/n$, is a very convenient tool [8]. Certainly, for singular integrands $f(n)$ more sophisticated analytic methods have to be employed (see Barton [14]). However, according to the EMF, in a long-distance expansion boundary corrections show up only at singularities of $f(n)$. Thus it is possible to isolate the potentially contributing terms at an early stage of the calculation and to obtain the results in a simple and transparent way.

A remark is in order, concerning possible contributions at infinity. Of course, for individually convergent sum and integral in (2.4) there should be no problem. The mass renormalisation, however, is divergent with only the difference D in (2.4) giving rise to a finite correction to the physical mass of the electron. Still worse, decomposition of $f(n)$ into regular and singular terms will yield even unbounded integrands $f_i(n)$ (with $f = \Sigma f_i$). Such problems have been treated extensively by Barton [14], who has given reasonable definitions of D by appropriate cutoff procedures. In (2.4) only the even part of $f(n)$ can contribute to D, so that, in the case of divergence, D can be defined by

$$D \equiv \lim_{\delta \to 0} \left(\sum_{n=0}^{\infty}{}' - \int_0^{\infty} \mathrm{d}n \right) \tfrac{1}{2}[f(n) + f(-n)] \exp(-\delta n) \tag{2.6}$$

where the prime indicates that we take $\frac{1}{2}$ of the $n=0$ term in the sum. For an at most linearly growing smooth function $f(n)$ it would be sufficient to simply give equal weight to the sum and integral, respectively, at the boundary.

Fischbach and Nakagawa [5, 6] have proposed a strong dependence of boundary corrections on the plasma frequency Λ at which a conductor becomes transparent for electromagnetic radiation. However, because boundary corrections are long-distance effects, such corrections should be at most of order [8] $h/\Lambda a \sim h \times 10^{-4}$. Again the EMF gives an estimate of the plasma frequency dependence to the end that only negligible higher-order corrections can arise for a smooth cutoff behaviour. Indeed, as pointed out by Tang [15], the strong cutoff dependence found in [5, 6] is due to the singular cutoff functions employed which do not meet the criteria given by Barton [14].

3. Electron mass and anomalous magnetic moment

We now turn to the calculation of the divergent one-loop graphs Σ and Λ of the self-energy and vertex correction, respectively (figure 2), which will eventually provide the boundary corrections to the physical quantities electron mass and anomalous magnetic moment. It will be important to verify the Ward identity $Z_1 = Z_2$ which guarantees that wavefunction renormalisation cannot change the magnetic moment associated with γ_μ in the on-shell decomposition of the current

$$j_\mu = \bar{u}(p-q)\left(\gamma_\mu f_1(q^2) - \frac{\mathrm{i}}{2m}\sigma_{\mu\nu}q^\nu f_2(q^2)\right)u(p) \qquad a_e \equiv g/2 - 1 \equiv f_2(0) \tag{3.1}$$

in the effective action.

3.1. Electron mass

First we calculate the matrix element of Σ between the localised wavepackets (2.3):

$$\begin{aligned}\langle\bar{u}|\Sigma|u\rangle &\equiv \int\!\!\int \mathrm{d}^4x\, \mathrm{d}^4y\, \bar{u}(y)\Sigma(x,y)u(x) \\ &= \mathrm{i}e^2 \int \mathrm{d}^4x \int \mathrm{d}^4y\, \bar{u}(y)\gamma^\mu \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{\exp[\mathrm{i}q(x-y)]}{\not{q}-m} \gamma^\nu u(x)\Delta_{\mu\nu}(x,y).\end{aligned} \tag{3.2}$$

Inserting (2.2) and (2.3) and performing the x, y and q integrations we obtain

$$\begin{aligned}\langle\bar{u}|\Sigma|u\rangle = \int \frac{\mathrm{d}p_3}{2\pi} \int \frac{\mathrm{d}p_3'}{2\pi} 2\sigma\pi^{1/2} \exp[-\tfrac{1}{2}\sigma^2(p_3^2+p_3'^2)] \frac{-\mathrm{i}e^2}{2a} \sum_{\substack{n=-\infty \\ k_3=n\pi/a}}^{\infty} \int \frac{\mathrm{d}^3\underline{k}}{(2\pi)^3} \\ \times \frac{\bar{u}(p')\gamma^\mu(\not{p}-\not{k}+m)\gamma^\nu u(p)}{k^2[(p-k)^2-m^2]}(2\pi)^4[g_{\mu\nu}\delta^4(p'-p) - \theta^n g_{\mu\bar{\nu}}\delta^4(p'-p+k-\bar{k})].\end{aligned} \tag{3.3}$$

The shift in p' from the second δ function (and the corresponding possible uncertainty in p' due to momentum transfer to the plates) is proportional to k_3. As we shall see presently, such terms yield regular 'integrands' $f(n)$ in D of (2.4) and can be omitted according to the discussion of the EMF above. Feynman parametrisation of the denominator then gives

$$\begin{aligned}\langle\bar{u}|\Sigma|u\rangle &= (2\pi)^3\delta^3(\underline{p}'-\underline{p}) \int \frac{\mathrm{d}p_3}{2\pi} 2\sigma\pi^{1/2} \exp(-\sigma^2p_3^2)\Sigma(p) \\ \Sigma(p) &= \frac{-\mathrm{i}e^2}{2a} \sum_{\substack{n=-\infty \\ k_3=n\pi/a}}^{\infty} \int \frac{\mathrm{d}^3\underline{k}}{(2\pi)^3} \int_0^1 \mathrm{d}x\, \bar{u}(p) \frac{(1-\theta^n)[4m-2(\not{p}-\not{k})]+2\theta^n(\bar{\not{k}}-\bar{\not{p}}+m)}{[(k-xp)^2-x^2p^2+x(p^2-m^2)]^2} u(p).\end{aligned} \tag{3.4}$$

The decomposition $g_{\mu\bar{\nu}}\gamma^\mu \ldots \gamma^\nu = \gamma^\alpha \ldots \gamma_\alpha - 2\gamma^3 \ldots \gamma_3$ has led to non-covariant terms proportional to $k_3\gamma^3$ (thus regular and negligible) or proportional to $p_3\gamma^3$. The latter, when weighted with the Gaussian distribution, are associated with powers of the small factor $1/\sigma m$ (according to such arguments the shift in k_3 in the denominator proportional to xp_3 can be disregarded as well). In the case of motion orthogonal to the plates the investigation is complicated by such anisotropic terms (see Barton [2]).

We proceed as in the usual analysis of on-shell mass and wavefunction renormalisation in momentum space, where δm and Z_1 are associated with the divergent terms Σ_0 and Σ_1 in the expansion

$$\Sigma(p)=\Sigma_0+(\not p-m)\Sigma_1+\ldots \tag{3.5}$$

(note that we have not yet used the on-shell condition). Regularisation of UV divergences is not necessary, because the k_3 integration is already regularised by (2.6) and $\delta\to 0$ therein commutes with optional further regularisations.

Putting the pieces together and performing the $\mathrm{d}^3\underline{k}$ integration

$$\begin{aligned}\Sigma_0&=\frac{e^2}{8\pi a}\sum_{n=-\infty}^{\infty}\int_0^1\mathrm{d}x\frac{(1+x)(1-\theta^n)+x\theta^n}{(x^2+n^2h^2)^{1/2}}\\&=-\frac{\alpha m}{2\pi}h\sum_{n=-\infty}^{\infty}\left[(1-\theta^n)\ln\left(\frac{1+(1+n^2h^2)^{1/2}}{|nh|}\right)+(1+n^2h^2)^{1/2}-|nh|\right]\end{aligned} \tag{3.6}$$

$$\begin{aligned}\Sigma_1&=\left(\frac{p_\mu}{m}\frac{\partial}{\partial p_\mu}\Sigma\right)\bigg|_{\not p=m}\\&=\frac{-\mathrm{i}e^2}{2a}\sum_{\substack{n=-\infty\\k_3=n\pi/a}}^{\infty}\int\frac{\mathrm{d}^3\underline{k}}{(2\pi)^3}\int_0^1\mathrm{d}x\left(\frac{-2}{(k^2-x^2m^2)^2}\right.\\&\qquad\left.+\frac{8x}{m}\frac{kp\not k-(1-x)(1+x-\theta^n)m^3}{(k^2-x^2m^2)^3}\right)\\&=\frac{\alpha}{2\pi}h\sum_{n=-\infty}^{\infty}\int_0^1\mathrm{d}x\left(\frac{1-x}{(x^2+n^2h^2)^{1/2}}-x\frac{1-x^2-\theta^n(1-x)}{(x^2+n^2h^2)^{3/2}}\right)\\&=\frac{\alpha}{2\pi}h\sum_{n=-\infty}^{\infty}\left[-\frac{1-\theta^n}{|nh|}+(1-\theta^n)\ln\left(\frac{1+(1+n^2h^2)^{1/2}}{|nh|}\right)\right.\\&\qquad\left.+(1+n^2h^2)^{1/2}-|nh|\right].\end{aligned} \tag{3.7}$$

The singular terms $1/nh$ and $\ln nh$ arose from the x integrations

$$\begin{aligned}&\int_0^1\mathrm{d}x\frac{1}{(x^2+n^2h^2)^{1/2}}=\ln\left(\frac{1+(1+n^2h^2)^{1/2}}{|nh|}\right)\\&\int_0^1\mathrm{d}x\frac{x+ax^2}{(x^2+n^2h^2)^{3/2}}=\frac{1}{|nh|}-\frac{1}{(1+n^2h^2)^{1/2}}\\&\qquad+a\left[\ln\left(\frac{1+(1+n^2h^2)^{1/2}}{|nh|}\right)+\frac{n^2h^2}{(1+n^2h^2)^{1/2}}-|nh|\right].\end{aligned} \tag{3.8}$$

Thus it is clear that the $k_3=n\pi/a$ terms, which were omitted above, would only have contributed to regular functions of n.

Finally, subtracting the infinite-space value of Σ_0, we obtain

$$\frac{\Delta m}{m}=\frac{\alpha}{\pi}h\left(\sum_{n=0}^{\infty}{}'[1-\cos(2\pi nt)]-\int_0^\infty\mathrm{d}n\right)(-\ln n)=\frac{\alpha}{4ma}[\psi(t)+\psi(1-t)+2C] \tag{3.9}$$

in terms of the ψ function and the Euler-Mascheroni constant $C=-\psi(1)$. Details of this calculation are given in the appendix. The further singular function $|n|$ in (3.6) is continuous, though not differentiable at $n=0$, and thus contributes only to order h^2.

The quantity $\Delta\Sigma_1$ is singular and has to be regularised with a photon mass. This is not surprising, because the wavefunction renormalisation is well known to be IR divergent on shell. However, we do not need its (regularised) value, but merely have to check that it cancels the analogous term $f_1(0)$ from the vertex correction to be calculated below.

3.2. Anomalous magnetic moment

The anomaly factor a_e and the vertex renormalisation form factor $f_1(0)$ are now calculated from the vertex correction graph figure 2(*b*):

$$\iiint_{x,y,z} \bar{u}(y)\Lambda_\mu(x,y;z)u(x)A^\mu(z)$$

$$= \iiint_{x,y,z} \bar{u}(y)\gamma^\alpha S_{\mathrm{F}}(y-x)\gamma_\mu S_{\mathrm{F}}(z-x)\gamma^\beta u(x)A^\mu(x)\Delta_{\alpha\beta}(y,x) \tag{3.10}$$

(S_{F} is the fermion propagator). Inserting the Fourier representations of all expressions and performing the configuration space integrations, we obtain

$$\iiint_{x,y,z} \bar{u}\Lambda_\mu u A^\mu = \int \frac{\mathrm{d}p_3}{2\pi}\int\frac{\mathrm{d}p_3'}{2\pi} 2\sigma\pi^{1/2}$$

$$\times \exp[-\tfrac{1}{2}\sigma^2(p_3^2+p_3'^2)]\bar{u}(p')\Lambda_\mu(p',p)u(p)A^\mu(p-p') \tag{3.11}$$

with

$$\Lambda_\mu(p-q,p) = \frac{-\mathrm{i}e^2}{2a}\sum_{\substack{n=-\infty\\ k_3=n\pi/a}}^{\infty}\int\frac{\mathrm{d}^3\underline{k}}{(2\pi)^3}\frac{(1-\theta^n)T(k)-\theta^n T_3(k)}{k^2[(p-q-k)^2-m^2][(p-k)^2-m^2]}$$

$$T(k) = (4m^2+2k^2)\gamma_\mu - 4(\not{k}+m)k_\mu - 2\gamma_\mu\not{q}\not{k} + 2\not{k}\not{q}\gamma_\mu \tag{3.12}$$

$$T_3(k) = -4\not{k}k_\mu + 2k^2\bar{\gamma}_\mu.$$

On the mass shell we now employ the Feynman parametrisation

$$\Lambda_\mu(p-q,p) = \frac{-\mathrm{i}e^2}{2a}\sum_{\substack{n=-\infty\\ k_3=n\pi/a}}^{\infty}\int\frac{\mathrm{d}^3\underline{k}}{(2\pi)^3}\int_0^1\mathrm{d}t\int_0^1\mathrm{d}x\, 2x\frac{(1-\theta^n)T(k)-\theta^n T_3(k)}{\{[(k-x(p-tq)]^2-x^2m^2\}^3} \tag{3.13}$$

shift the k integration and perform the trivial t integration. Retaining only terms which eventually yield singular functions of n, we arrive at

$$\frac{-\mathrm{i}e^2}{2a}\sum_{\substack{n=-\infty\\ k_3=n\pi/a}}^{\infty}\int\frac{\mathrm{d}^3\underline{k}}{(2\pi)^3}\int_0^1\frac{\mathrm{d}x\,2x}{(k^2-x^2m^2)^3}(1-\theta^n)[4\gamma_\mu(1-x)m^2+2\mathrm{i}xm\sigma_{\mu\nu}q^\nu]. \tag{3.14}$$

Thus, to first order in h, the boundary corrections are to be calculated from

$$\begin{aligned} f_1(0) &= \frac{\alpha}{2\pi}h\sum_{n=-\infty}^{\infty}(1-\theta^n)\left[-\frac{1}{|nh|}+\ln\left(\frac{1+(1+n^2h^2)^{1/2}}{|nh|}\right)+\text{regular terms}\right]\\ f_2(0) &= \frac{\alpha}{2\pi}h\sum_{n=-\infty}^{\infty}(1-\theta^n)\left[\ln\left(\frac{1+(1+n^2h^2)^{1/2}}{|nh|}\right)+\text{regular terms}\right]. \end{aligned} \tag{3.15}$$

The vertex correction form factor $f_1(0)$ exactly cancels the wavefunction renormalisation (3.7). This is required by consistency only for the 'IR-singular' part $1/|nh|$. However, since all anisotropic terms could be neglected, the formula

$$\Sigma_1 = \frac{p_\mu}{m} \frac{\partial}{\partial p_\mu} \sum(p)|_{\not p = m}$$

guarantees the Ward identity $Z_1 = Z_2$ to hold up to higher-order terms. Hence, there are no boundary corrections to the Dirac magnetic moment associated with γ_μ. Finally, the correction to the anomaly factor $a_e = f_2(0)$ equals the correction to the mass counterterm given in (3.9):

$$\Delta a_e = \frac{\alpha}{4ma}[\psi(t) + \psi(1-t) + 2C]. \tag{3.16}$$

This coincidence has an important effect. Since the magnetic moment of the electron $\mu = g^{e/m}$ depends on the ratio $(1 + a_e)/m$, the boundary corrections to the 'ordinary' and the 'anomalous' magnetic moment *exactly cancel*. At one loop and to first order in h there is no boundary correction to the (total) magnetic moment; the gyromagnetic ratio and the electron mass are both decreased by the same factor. This observation resolves the apparent discrepancy between the results of Brown *et al* [9-11], who predicted no boundary correction to the spin-flip frequency in the geonium experiment, and the first-order corrections to the anomaly factor, predicted by various other groups [3-8].

4. Summary and discussion

In this paper we have performed a consistent treatment of renormalised QED between conducting plates at one loop and to first order in a long-distance expansion in $1/ma$ (the distances to the plates are d and $a-d$, respectively, and we use natural units). The boundary corrections to wavefunction renormalisation and photon vertex γ_μ, though IR divergent (and gauge dependent), cancel according to the Ward identity $Z_1 = Z_2$. In the case of motion parallel to the plates both physical parameters of QED, the electron mass m and its anomaly factor a_e, are decreased by the boundary corrections

$$\Delta a_e = \frac{\Delta m}{m} = \frac{\alpha}{4ma}[\psi(t) + \psi(1-t) + 2C] \qquad t = d/a. \tag{4.1}$$

In agreement with the argument of Brown *et al* [10, 11] it turns out, by reinserting Planck's constant, that these results are classical. In fact, Δm is exactly the electrostatic energy of the electron due to its mirror charges†. The higher-order corrections which we have neglected are associated with powers of $\hbar$. Recently boundary corrections have been calulated for plane wave electron wavefunctions [5, 7, 8]. We recover their leading logarithmic term $-(\alpha/2ma)\ln(a/a_0)$ by averaging our result (4.1) over t with some cutoff distance a_0. In the opinion of the author the precise location of the cutoff

† I am grateful to Barton for drawing my attention to this fact and for informing me about his related work prior to publication.

is a rather academic question as long as no experimental situation is specified (it appears hard to realise plane waves extending to the surface of the conductor).

Due to broken translational invariance we had to set up our calculation in configuration space. Nevertheless it turned out that, in the leading approximation, the whole analysis could be performed within momentum space. This is intuitively clear from the uncertainty relation: the commutator of the momentum operator p_z with z-dependent corrections proportional to $1/(a-z)$ is of order $1/(a-z)^2$. Thus, to first order in $1/ma$, localisation of the electrons is consistent with a well defined momentum. The calculation of higher-order corrections, which are genuine quantum mechanical, will require a more involved concept of the localisation of the electrons.

As special cases we give the results for electrons localised in the middle between the plates or approaching one of the plates:

$$\Delta a_e = \frac{\Delta m}{m} = \begin{cases} -(a/ma) \ \ln 2 & d = a/2 \\ -\alpha/4md & d \ll a. \end{cases} \tag{4.2}$$

The latter case $a \to \infty$ provides us with the boundary correction due to a single conducting plate at distance d.

In view of a controversy in the recent literature [3-11] the coincidence of the corrections to the gyromagnetic ratio and electron mass, respectively, is of particular interest. Whereas most authors found a correction to a_e of order α/ma, Brown *et al* [10, 11] could show that there is no correction of this order to the spin-flip frequency $\omega_s = geB/2mc$ in the geonium experiment (there is, on the other hand, a correction to the cyclotron frequency $\omega_c = eB/mc$ [11]). On this basis they conjectured errors in previous work [4, 7] due to lack of gauge independence. However, the manifestly gauge-independent calculation of Kreuzer and Svozil [8] confirmed the first-order results. This apparent discrepancy is now trivially resolved by the coincidence of $\Delta m/m$ and Δa_e: the respective corrections to the 'ordinary' and the 'anomalous' magnetic moment of the electron exactly cancel to order α/ma. The authors of [3-8] did not distinguish correctly between the anomaly factor and the magnetic moment. Since we have only treated the weak-field limit our results are certainly not applicable to geonium, where the cyclotron frequency of the electron is of the order of the lowest electromagnetic cavity modes and resonances are to be expected [11, 12]. However, our results indicate that a similar cancellation might take place in the experimental setup considered by Brown *et al* [11].

Finally we insert some reasonable parameters in order to obtain an estimate of the magnitude of boundary corrections. For electrons localised in the middle between conducting plates at a distance of 1 cm

$$\Delta a_e = \Delta m/m = 2\times 10^{-13} \qquad a = 2d = 1 \text{ cm}. \tag{4.3}$$

The correction to the magnetic moment is of order $\alpha/(ma)^2$, and is thus smaller by ten orders of magnitude, as had been estimated by Brown *et al* [10]. Since resonances in the geonium experiment could increase this effect considerably, further improvement of the extraordinary precision of recent experiments [9]

$$a_e^{\exp} = 1.001\,159\,652\,193\,(4) \tag{4.4}$$

may lead to at least a qualitative verification of boundary corrections due to conducting surfaces.

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Appendix

In this appendix we calculate the difference D between the sum and integral (see (2.6)) for the function

$$f(n)=\ln|n|[1-\cos(2n\pi t)] \tag{A1}$$

which is the only singular term contributing to Δm and Δa_e in (3.9).

To this end we use the Abel–Plana formula [14]

$$D[f(n)]=\mathrm{i}\int_0^\infty \frac{\mathrm{d}z}{\exp(2\pi z)-1}[f(\mathrm{i}z)-f(-\mathrm{i}z)]. \tag{A2}$$

This formula is found by expressing the sum $\Sigma f(n)$ as a contour integral of $f(n)\cot(2\pi n)$ and deforming the contour to the imaginary axis. Thus $f(z)$ has to be analytic without singularities in the positive complex half-plane and the contour integral along the half-circle at infinity has to be checked to vanish.

Both criteria are fulfilled by the function given above if $|t|<1$, so that [16]

$$\begin{aligned}\left(\sum_{n=0}^{\infty}{}' - \int_0^\infty \mathrm{d}n\right)\ln(n)[1-\cos(2\pi nt)] \\ &= \mathrm{i}\int_0^\infty \frac{\mathrm{d}z}{\exp(2\pi z)-1}[\ln(\mathrm{i}z)-\ln(-\mathrm{i}z)][1-\cosh(2\pi zt)] \\ &= \frac{1}{4}\int_0^\infty \mathrm{d}z\frac{\exp(zt)+\exp(-zt)-2}{\exp(z)-1} = -\tfrac{1}{4}[\psi(t)+\psi(-t)+2C]\end{aligned} \tag{A3}$$

where $C=-\psi(1)$ denotes the Euler–Mascheroni constant. At first sight this result seems to be wrong, since the original symmetry $d\to a-d$ (i.e. $t\to 1-t$) of the configuration in figure 1 is no longer present. But one has to be careful: although the Fourier (cosine) transform $\int_{-\infty}^{\infty}\mathrm{d}x\,\ln|x|\cos(2\pi tx/h)$ vanishes in the limit $h\to 0$ (we have to recover the infinite-space result), this term is of order $O(h)$! In other words: we have to subtract the infinite-space value $\int_0^\infty \mathrm{d}n\,\ln(n)$ instead of $\int_0^\infty \mathrm{d}n\,\ln(n)[1-\cos(2\pi nt)]$. The additional contribution

$$\int_0^\infty \mathrm{d}n\,\ln(n)\cos(2\pi nt)=-1/4t \tag{A4}$$

(obtained by analytic continuation of the Laplace transform [16] of $\ln n$) with $\psi(1+t)=\psi(t)+1/t$ just restores the symmetry $t\to 1-t$:

$$\left(\sum_{n=0}^{\infty}{}'[1-\cos(2n\pi t)]-\int_0^\infty \mathrm{d}n\right)\ln n=-\tfrac{1}{4}[\psi(t)+\psi(1-t)+2C]. \tag{A5}$$

Of course, all divergent expressions above have to be regularised by a factor $\exp(-\delta n)$ with $\delta\to 0+$ at the end of the calculations.

References

[1] Casimir H B G 1948 *Proc. K. Neder. Akad. Wet.* B **51** 793
Balian R and Duplantier B 1978 *Ann. Phys., NY* **112** 165
Brown L S and Maclay G L 1969 *Phys. Rev.* **184** 1272
Bordag M, Robaschik D and Wieczorek E 1985 *Ann. Phys., NY* **165** 192
Plunien G, Müller B and Greiner W 1986 *Phys. Rep.* **134** 87
[2] Barton G 1970 *Proc. R. Soc.* A **320** 251
Babiker M and Barton G 1972 *Proc. R. Soc.* A **326** 255
[3] Babiker M and Barton G 1972 *Proc. R. Soc.* A **326** 277
[4] Barton G and Grotch H 1977 *J. Phys. A: Math. Gen.* **10** 1201
[5] Fischbach E and Nakagawa N 1984 *Phys. Lett.* **149B** 504; 1984 *Phys. Rev.* D **30** 2356
[6] Fischbach E and Nakagawa N 1985 *Preprint* Purdue University PURD-TH-85-9
[7] Svozil K 1985 *Phys. Rev. Lett.* **54** 742
[8] Kreuzer M and Svozil K 1986 *Phys. Rev.* D **34** 1429
[9] Brown L S and Gabrielse G 1986 *Rev. Mod. Phys.* **58** 233
[10] Boulware D G, Brown L S and Lee T 1985 *Phys. Rev.* D **32** 729
[11] Brown L S, Gabrielse G, Helmerson K and Tan J 1985 *Phys. Rev. Lett.* **55** 44
[12] Bordag M 1986 *Phys. Lett.* **171B** 113
[13] Olver F W J 1974 *Asymptotics and Special Functions* (New York: Academic)
[14] Barton G 1981 *J. Phys. A: Math. Gen.* **14** 1009
[15] Tang A C 1987 *Phys. Rev.* D **36** 2181
[16] Gradshteyn I S and Ryzhik I M 1980 *Tables of Integrals, Series, and Products* (New York: Academic)